An illustrated guide to IUH/GIUH estimation

Riccardo Rigon[1], Silvia Franceschi[2], Giuseppe Formetta[1], Marialaura Bancheri[3], and Niccolò Tubini[1]

[1]University of Trento
[2]Hydrologis
[3]ISAFOM

January 11, 2022

Abstract

This is an informal treatment of the instantaneous unit hydrograph IUH/GIUH (Rigon et al., 2015) estimation under the simplifying assumption that rainfall inputs are given as a discrete sequence of rain impulses (that we will call from now, rain record) This, more than being a simplification, is related to the discrete nature of rain at small time scales of aggregation and has a physical basis (e.g. (Lovejoy & Mandelbrot, 1985)). This manuscript is distributed according to the CC v4 license.

Let's have a record of precipitation of duration $\Delta t \ll t_c$ where t_c is the *concentration time* (Rigon et al., 2011) of the catchment we are analyzing (t_c can be thought as the length of the domain over which the IUH is defined or a conventional time where the contribution to the discharge given by a particular rainfall record is not anymore detectable). Let t be the clock time and t_{in} the time at which the precipitation record happens. Finally let $p_{t_{in}}$ the precipitation intensity (for instance in mm/h, [L/T]) during the time interval $(t_{in}, t_{in} + \Delta t]$

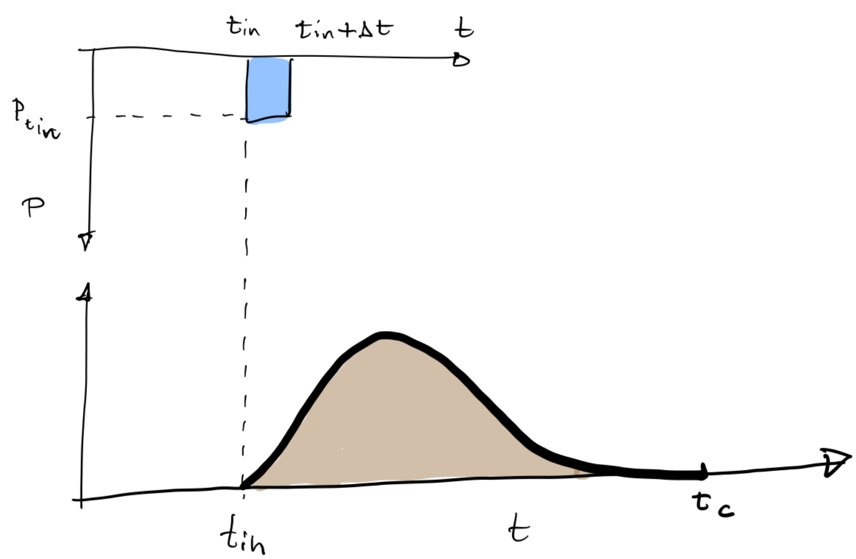

Figure 1: Representation of the rainfall record described in the text.

The precipitation is well represented in Figure 1 above. In this case, according to the IUH theory, the discharge at any time t is given by:

$$q(t, t_{in}) = A\, p_{t_{in}} \Theta(t_{in}) \int_{t_{in}}^{t} f(t-x) H(x) H(t-(x+\Delta t)) dx \quad (1)$$

where:

- $q\,(t, t_{in})$ $[\mathrm{m^3 s^{-1}}]$ is the discharge generated by the single rain record called *age-ranked discharge*.
- $f\,(t-t_{in})$ $\left[s^{-1}\right]$ is the IUH (here assumed time invariant with respect to the injection time)
- $A\left[m^2\right]$is the area of the catchment we are analyzing
- $H\,(x)$ is the Heaviside step function such that

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases} \quad (2)$$

- $\Theta\,(t_{in})$ is a coefficient of partition that identifies which portion of the rain input (at time t_{in} goes into discharge). $\Theta\,(t_{in}) \le 1$ depending if other outputs (as evapotranspiration) are present, but in the present case, for simplicity, will assume $\Theta\,(t_{in}) = 1$

For simplicity, we will assume in the following that A is included in f definition, which now become $\left[m^2 s^{-1}\right]$. The notation used above is congruent with the one used in (Rigon et al., 2016) where, however, more general cases are treated.

According to the definition of the Heaviside step function the integral giving the age-ranked discharge can be considered for various time steps.

- For $t < t_{in}$ is clearly:

$q\,(t, t_{in}) = 0$

- For $t_{in} < t \le t_{in} + \Delta t$:

$q\,(t, t_{in}) = p_{t_{in}} \int_{t_{in}}^{t} f\,(t-x)\, dx$

because $H\,(x)\, H\,(t-x) \equiv 1$

- For $t > t_{in} + \Delta t$:

$q\,(t, t_{in}) = p_{t_{in}} \int_{t_{in}}^{t_{in}+\Delta t} f\,(t-x)\, dx$

because $H\,(x)\, H\,(t-x) \equiv 1$ for $t_{in} < x \le t_{in} + \Delta t$ and $H\,(x)\, H\,(t-x) \equiv 0$ for $t > t_{in} + \Delta t$.

Let's now define the so called S-function (which, as it is known, correspond to the probability or residence/travel times (Rigon et al., 2016), in this case multiplied by the catchment area A:

$$s_f(u) := \int_0^u f(y) dy \quad (3)$$

we can show that the previous integral can be expressed as function of $s_f\,(\,)$ which is convenient when it is known in advance (for instance by assuming or deriving it from some geomorphic procedure (Rigon et al., 2015)).

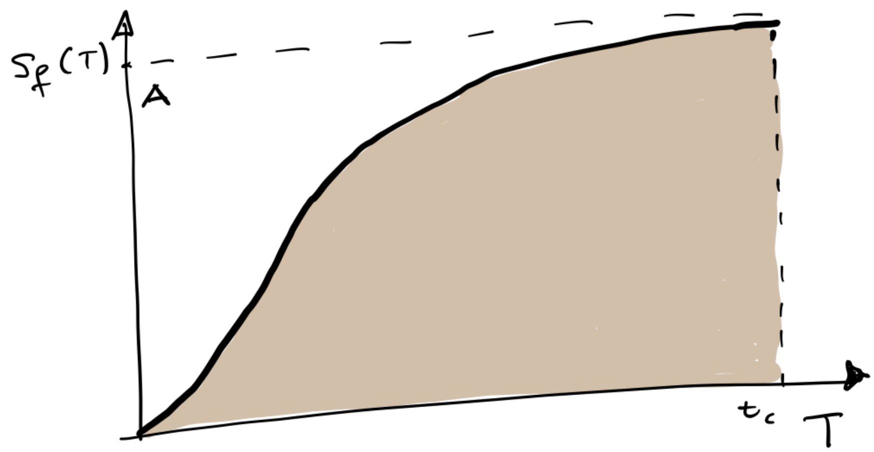

Figure 2: A pictorial representation of the function $s_f\,(T)$. After the inclusion of the catchment area in the definition, it varies from 0 to the maximum value of A.

Let's then define the travel time Tas the difference between the clock time and the injection time:

$$T := t - t_{in} \tag{4}$$

and use the integrating variable $z \;=\; t - t_{in}$, so that, $t_{in} < t \;\leq t_{in} + \Delta t$ becomes $0 \;<\; T \;\leq \Delta t$, $dz = -dx$ and the extremes of integration move such $x = t_{in} \;\; \rightarrow z = t - t_{in} \;= T$ and $x = t \;\rightarrow z = 0$:

$$q(t, t-T) = p_{t-T} \int_0^{t-t_{in}=:T} f(z)dz =: p_{t-T} s_f(T) \tag{5}$$

therefore, the age-ranked discharge s-function, in this case depends only on the travel time, nor on the injection time, neither on the clock time alone while the dependance of the injection time is retained only in the intensity of precipitation. Analogously $t \;>\; t_{in} + \Delta t$ becomes $T \;>\; \Delta t$ and then $x = t_{in} \;\rightarrow z = t - t_{in} \;= T$ and $x = t_{in} + \Delta t \;\rightarrow z = t - t_{in} - \Delta t \;=\; T \;-\; \Delta t$ while $dz \;=\; -dx$ such that:

$$q(t, t-T) = p_{t-T} \int_{T-\Delta t}^{T} f(z)dz = p_{t-T}\left(s_f(T) - s_f(T - \Delta t)\right) \tag{6}$$

This shows that for getting the age ranked discharges for any time, t, we need the values of $s_f\,(T) \quad \forall T$. Please also observe that, for any $t \;\; \geq t_c + \Delta t$,- it is $(s_f(T) - s_f(T - \Delta t)) \;= 0$ and therefore $q\,(t, t-T) \equiv 0$, as expected. To illustrate better the topic let's rework Figure 1 into Figure 3 below.

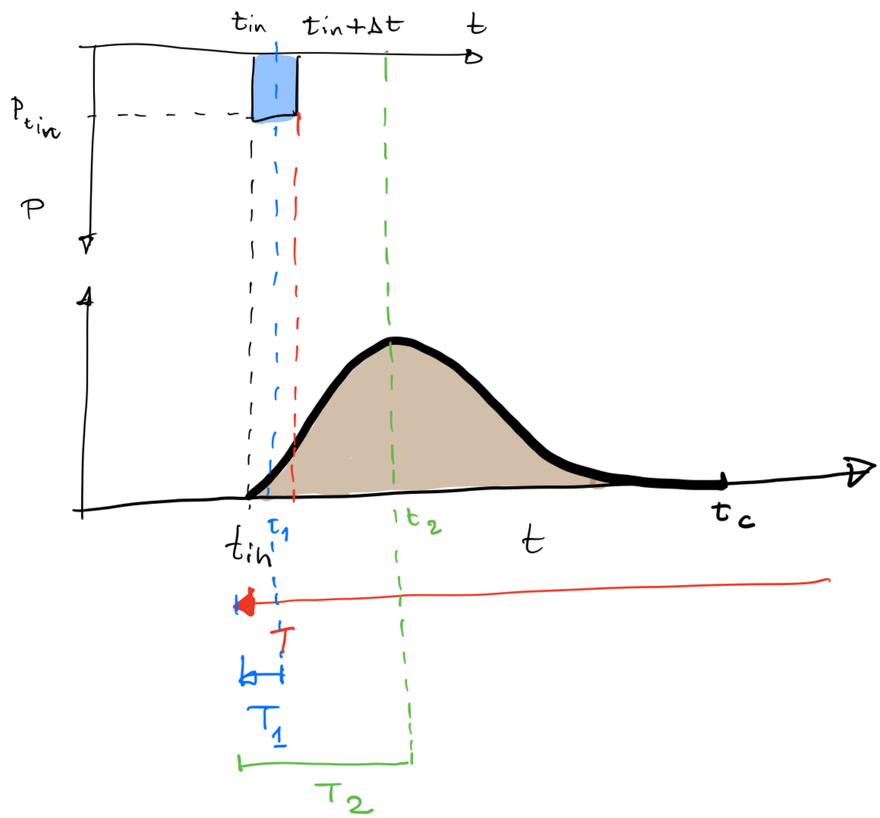

Figure 3: We select two time instant, t_1 and t_2 and, in the text below, we describe how to get the corresponding discharge.

Figure 2 shows the new axes represented by T. It goes in the opposite direction of t. For $t = t_1$ we are in the case $T < \Delta t$. Therefore the solution is given by (5). However, because t_1 is fixed, the extremes of the integral vary just between 0 and T_1. We can repeat the procedure for any T_1 in this interval. Upon the knowledge of f, the s-function is known in advance and we have just to pick its values at T_1 (and the values of the precipitation at $t_2 - T_2$) to know the discharge, as shown in Figure 4 below.

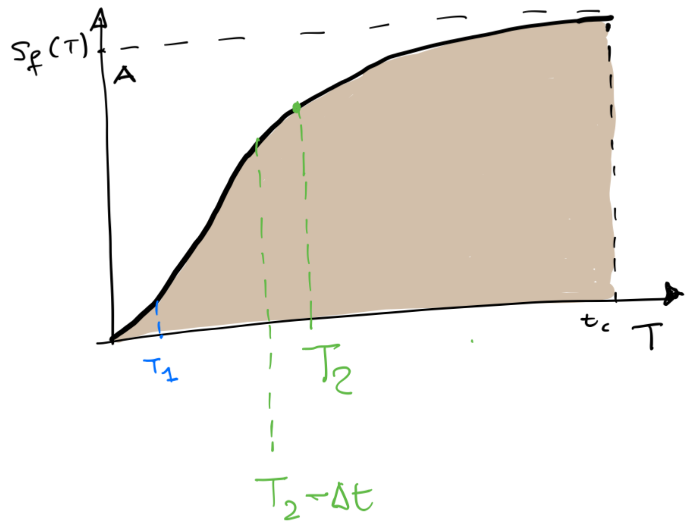

Figure 4: A pictorial representation od the s_f function with two time instants highlighted.

For T_2, since, it is $T_2 > \Delta t$, we have instead to take the difference of the s-function between the two instant $T_2 = t_2 - t_{in}$ and $T_2 - \Delta t = t_2 - t_{in} - \Delta t$.

Please observe that in the previous derivation Figure 3 can be misleading. In fact the integration in equation 1 is over t_{in} while Figure 3 represent the IUH vs clock time t. Therefore, in principle, our integral is not the marron area below the IUH (it is just in a special case, which is actually the one we are dealing with) and the change of variable we did (exchanging t_{in} with the travel time T) hides this fact. A right representation of the integration is, instead shown in Figure X below.

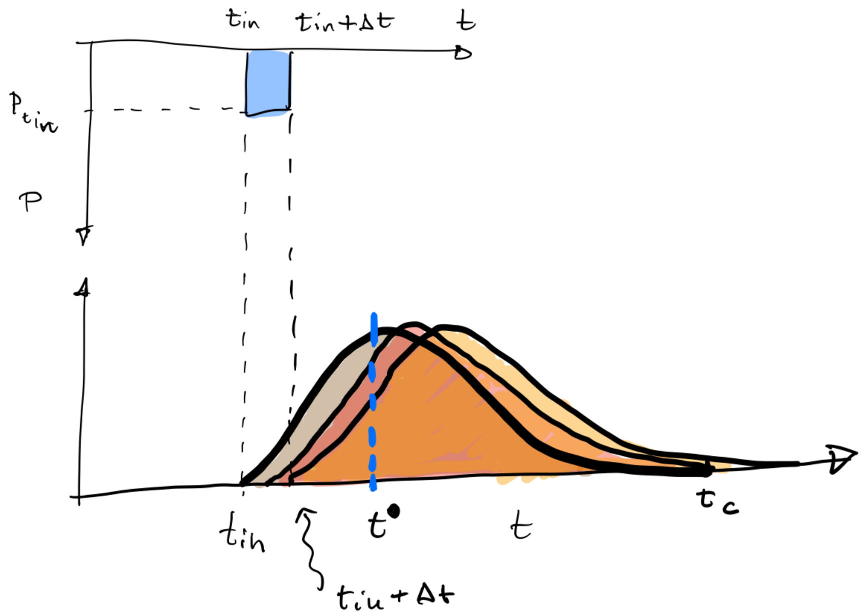

Figure 5: When integrating the IUH for obtaining the discharges we are integrating over t_{in} not overt t. This mean we have to consider a set of (infinitely many) f, each one for any available t_{in}. We actually integrate for any t the intersection of these curves with the plane (q, t_{in})

In Figure 5 we show three of the (infinite) curves that correspond to an instant of rainfall between t and $t + \Delta t$. For any $t = t^{\bullet}$ we actually integrate over these curves, not just one of them. The procedure we followed can be used to show that any of these sections in the third dimension is actually equal to the forefront maroon curve. But this is the case because we assumed $f = f(t - t_{in})$, depending just on the difference between the clock and injections time. In the most general case, as seen in (Rigon et al., 2016) , any of the f is time dependent, i.e. $f = f(t, t_{in})$ and neglecting the right integration direction can drive to conceptual mistakes easily.

Examples

The exponential travel time distribution

For making clearer the example let's consider first the simple case of

- $t_{in} = 0$, *i.e.* $T = t$
- $p_{t_{in}} = p$

and of:

$$f(t - 0) = \frac{1}{\lambda} e^{-(t-0)/\lambda} \tag{7}$$

where λ is a parameter that can be proven to be the mean travel time. Therefore, by integration is easy to prove that:

$$s_f(T) = A\left(1 - e^{-T/\lambda}\right) \tag{8}$$

Therefore for any $T \ \leq \Delta t$ (i.e. $t \ \leq \Delta t$), it is

$$q(t,0) = q(T=t) = A\,p\left(1 - e^{-t/\lambda}\right) \tag{9}$$

For $t \ > \ \Delta t$ is instead:

$$q(t,0) = A\,p\left(1 - e^{-t/\lambda}\right) - \left(1 - e^{-(t-\Delta t)/\lambda}\right) = A\,p\left(e^{-(t-\Delta t)/\lambda} - e^{-t/\lambda}\right) \tag{10}$$

Studying the function it is easy to show that the discharge has a maximum at $t = \Delta t$ equal to $A \ p \ \left(1 - \ e^{-\frac{\Delta t}{\lambda}}\right)$ and then decrease asymptotically to 0. In this case the concentration time is not defined but for any practical purpose this can be set when $t \ \geq \lambda \ \ln\left(\frac{1}{\alpha}\left(e^{\frac{\Delta t}{\lambda}} - 1\right)^{-1}\right)$ with $\alpha >> 1$. The discharge qualitative behavior is shown in Figure 5 below.

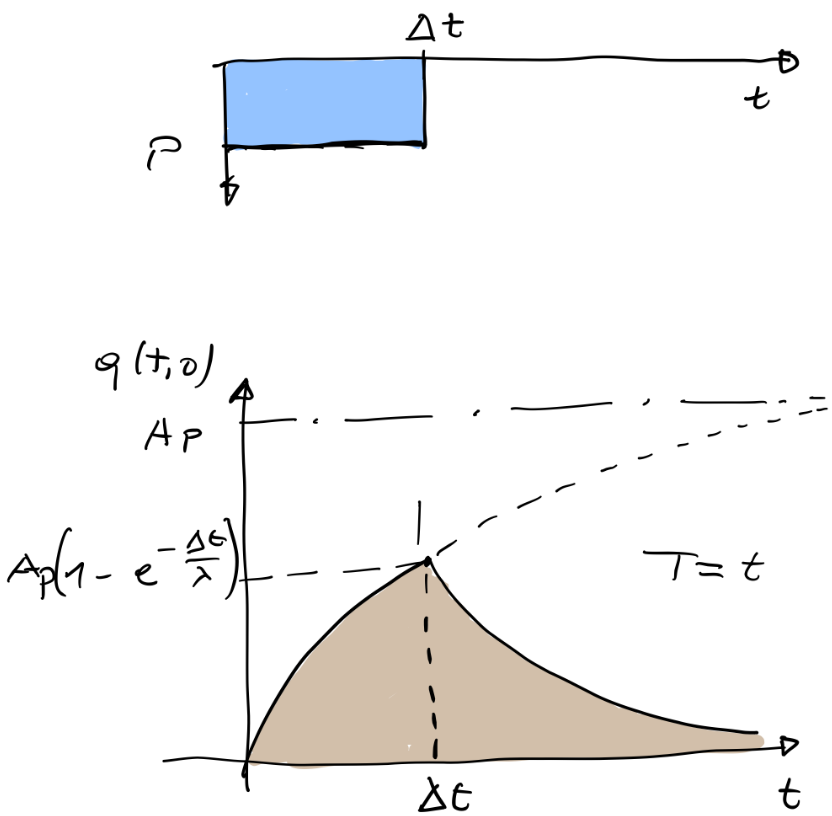

Figure 6: The behavior of the exponential IUH for a rainfall record at $t_{in} = 0$

The generalization to cases for which $t_{in} \neq 0$ is left for exercise.

The width function case

Another interesting and classical case is the case in which the IUH is given by a width function (Rigon et al., 2015). This is defined as the area of the catchment at a certain distance from the outlet measured along the drainage directions. A width function is usually derived from the analysis of a digital elevation model (DEM) and given at discrete time distance steps, according to the resolution of the DEM. To obtain from it a IUH, the geomorphic width function has to be properly normalised by the catchment area and space (distance to outlet) has to be mapped to time. Procedures to do it are illustrated, for instance in ("The uDig Spatial Toolbox for hydro-geomorphic analysis", 2014) and in (Rigon et al., 2015). The resulting IUH looks like the one in Figure 7 below:

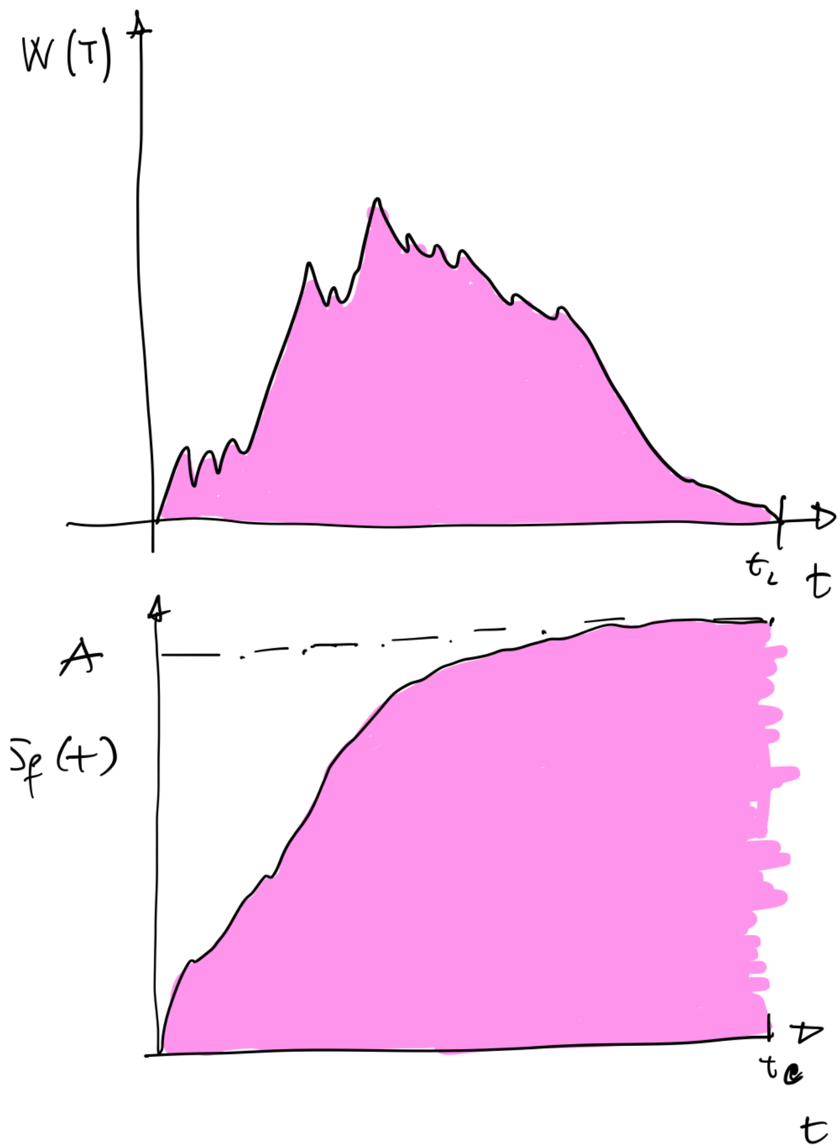

Figure 7: A pictorial representation of a width function and the corresponding s-function.

Once we get the s_f (remind that here t stands for T) that can be considered continuous when interpolated appropriately, the previous theory applies verbatim.

Numerics

We have now all the tools for implementing the numerics of the IUH, at least in the case of a single rain record. The main choice to do is to select a discretization of the time axis. A reasonable choice is to select a $\delta t \ < \ \Delta t$ and such not to loose the information derived from the f function, especially if it is given as a discrete set of points. In this case the continuous variables are set to:

- $t \ = \ j \ \delta t \quad j = 0, \ ..., \ n$ with $n \ \delta t \ = \ \Delta t$
- $t_{in} = i \ \delta t$
- $T \ = t - t_{in} = (j - i) \, \delta t$
- $t_c = m \ \delta t$

With these choices, equations 5 and 6 becomes:

$$q_{j,i} = p_i \begin{cases} s_f|_{j-i} & 0 < j - i \leq n \\ (s_f|_{j-i} - s_f|_{j-i-n}) & j - i > n \end{cases} \tag{11}$$

where $s_f \, |_k|$ is the s-function estimated at the $k - est$ point.

Examples

To illustrate the example let's consider the simple case in Figure 8 below.

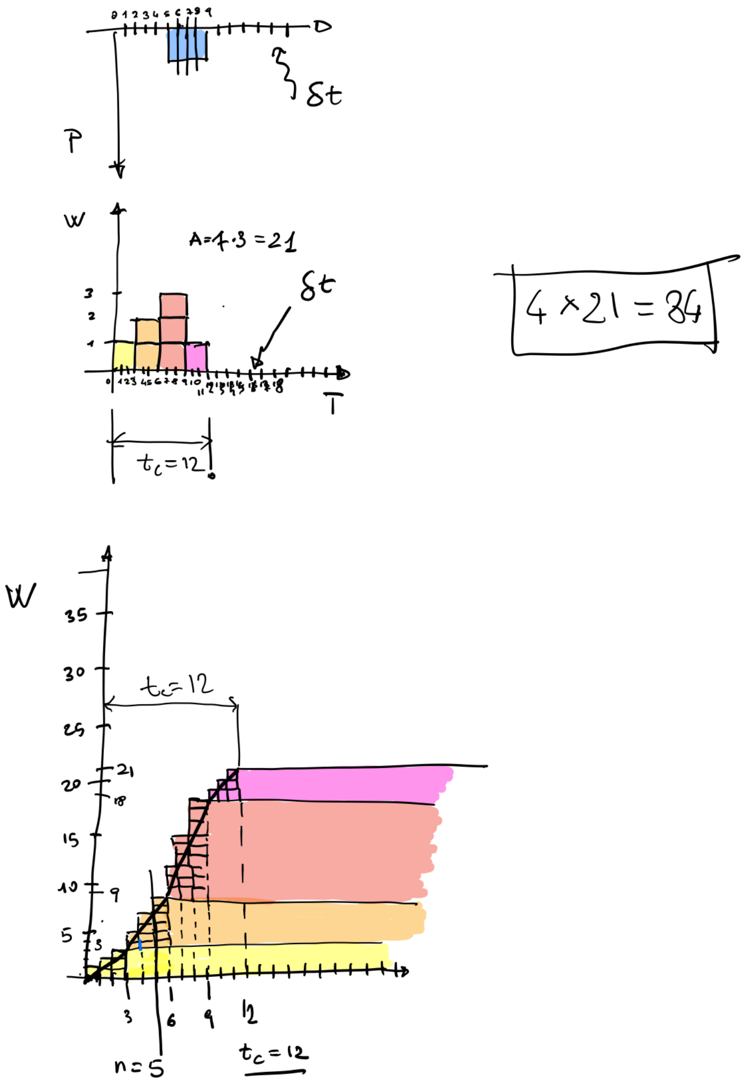

Figure 8: We consider here an example where there is a single record of precipitation at time $i = 5$ of duration $n = 4$ time steps (on top). The IUH is represented instead in the central figure. It is given over four time intervals, each one of 3 time steps. The total rainfall of the catchment is A times the volume of total rainfall $p_i \ \Delta t = p_i \ n \ \delta t$ where $n = 4$ is the number of intervals over which the rainfall duration spans. $A = 7 \ h = 21$ being $h = 3$ the number of intervals in which each IUH column is subdivided. On bottom there is the s-function resulting. The whole mass of the rainfall is then 84 p_i. The concentration time of the example is $t_c = 12$

The caption of the Figure explains all what is expected. To be specific the rainfall record has a total volume of 84 p_i. It is easy to estimate for all discrete times the value of the s-function as

$$s_f|_{j,i} = \{(0,0),(1,1),(2,2),(3,3),(4,5),(5,7),(6,9),(7,12),(8,15),(9,18),(10,19),(11,20),(12,21)\}$$

for any $t \ > 12 \ \delta t$ the s-function is not increasing anymore, i.e. $s_f\,(t) = 12$. For any $t \ \leq 5 \ \delta t$, $s_f\,(t) \ = 0$.

$q_{1,5} = p_5 s_f(-4) = 0$ $\quad$ $q_{2,5} = p_5 s_f(-3) = 0$ $\quad$ $q_{3,5} = p_5 s_f(-2) = 0$
$q_{4,5} = p_5 s_f(-1) = 0$ $\quad$ $q_{5,5} = p_5 s_f(0) = 0$ $\quad$ $q_{6,5} = p_5 s_f(1) = 1p_5$
$q_{7,5} = p_5 s_f(2) = 2p_5$ $\quad$ $q_{8,5} = p_5 s_f(3) = 3p_5$ $\quad$ $q_{9,5} = p_5 s_f(4) = 5p_5$
$q_{10,5} = p_5(s_f(5) - s_f(1)) = 6p_5$ $\quad$ $q_{11,5} = p_5(s_f(6) - s_f(2)) = 7p_5$ $\quad$ $q_{12,5} = p_5(s_f(7) - s_f(3)) = 9p_5$
$q_{13,5} = p_5(s_f(8) - s_f(4)) = 10p_5$ $\quad$ $q_{14,5} = p_5(s_f(9) - s_f(5)) = 11p_5$ $\quad$ $q_{15,5} = p_5(s_f(10) - s_f(6)) = 10p_5$
$q_{16,5} = p_5(s_f(11) - s_f(7)) = 8p_5$ $\quad$ $q_{17,5} = p_5(s_f(12) - s_f(8)) = 6p_5$ $\quad$ $q_{18,5} = p_5(s_f(13) - s_f(9)) = 3p_5$
$q_{19,5} = p_5(s_f(14) - s_f(10)) = 2p_5$ $\quad$ $q_{20,5} = p_5(s_f(15) - s_f(11)) = p_5$ $\quad$ $q_{21,5} = p_5(s_f(16) - s_f(12)) = 0$

Therefore

The table above represents all the estimable points. Taking the sum of all the values, it does 84 p_i as requested, meaning the the whole water in input exit the catchment. The table values are represented in Figure 9 below

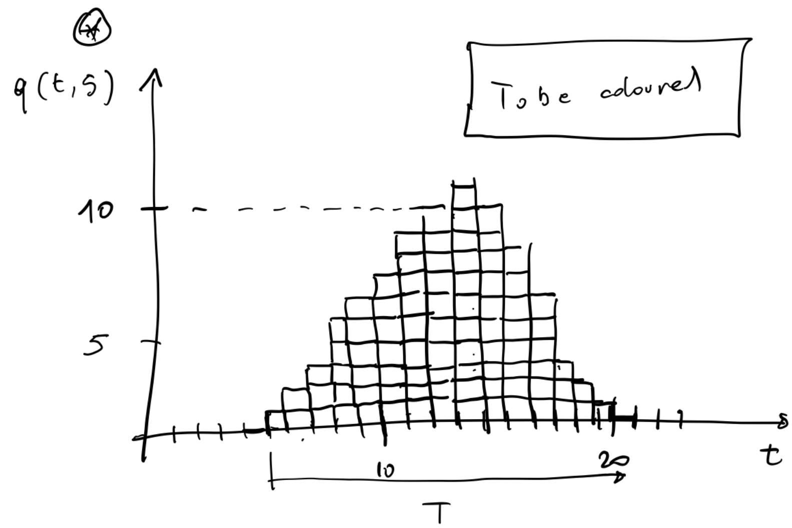

Figure 9: This is the hydrograph derived from the previous rainfall record. Still it attends to be coloured yellow, orange, red and violet to associate each discharge to its position in the IUH.

The case of multiple rainfall records

Now, let's consider the case of multiple rainfall records, as those shown in Figure Z. For simplification, we assume that each rainfall records is of the same duration, $n\delta t = 4\ \delta t$ where δt is the unit of time as in the examples above.

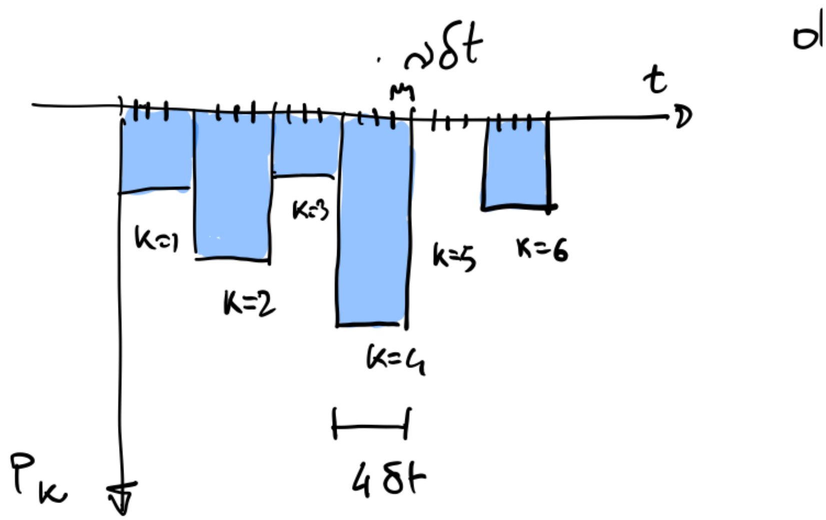

Figure 10: The case of multiple rainfall recordss of duration $n\delta t = 4\ \delta t$. As one can note there can be empty rainfall intervals ($p_k = 0$) without loss of generality.

It is a consequence of the IUH mathematical structure that each one of the records can be treated separately and the total discharge obtained by summing over all the rainfall records (we skip the proof though). The general expression for the discharge is then:

$$Q(t) = \sum_{k=0}^{l} p_{i(k)} \begin{cases} 0 & t < t_{in}(k) \\ s_f(t - t_{in}(k)) & t - t_{in}(k) \leq \Delta t \\ s_f(t - t_{in}(k)) - s_f(t - t_{in}(k) - \Delta t) & t - t_{in}(k) > \Delta t \end{cases} \tag{12}$$

with $l := \frac{t}{\Delta t}$ and where $i\,(k) = (k-1)\,n$ if $\Delta t = n\ \delta t$ and $t_{in}\,(k) = i\,(k)\ \ \delta t$. It can be verified that $Q\,(t) = 0$ for $t \geq t_c$. The discrete representation of the above equation is derived then as:

$$Q_j = \sum_{k=0}^{l} p_{i(k)} \begin{cases} 0 & j < i(k) < (k-1) * n \\ s_f|_{j-i(k)} & j - i(k) \leq n \\ s_f|_{j-i(k)} - s_f|_{j-i(k)-n} & j - i(k) > n \end{cases} \tag{13}$$

From the numerics point of view, there are just minimal changes from what happens in the case of a single rain record, with the addition of an iteration over the rain impulse and a different evaluation of $i\,(k)$s, at least when the IUH is not changing shape over time.

In this case it is clear that the domain where the hydrograph due to a specific rain record k is different from zero in a domain of length $t_c + \Delta t = (m+n)\ \delta t$. The total hydrograph, at any clock time t, is easily seen to be of length $(l \cdot n + m)\ \delta t$. Please notice that at any new record incoming the knowledge of future discharges due only to past rain record is available for all the m future time steps, being no modified by future at all.

In a real time perspective, when we receive a new rain record (of duration $n \cdot \delta t$) we have to expand the whole hydrograph domain by m steps forward in time.

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