Dirichlet Generating Functions

Benedict Irwin[1]

[1]University of Cambridge

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Well known Dirichlet series include:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{1}$$

$$\frac{1}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \tag{2}$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\lambda(k)}{k^s} \tag{3}$$

If we define the inverse Dirichlet transform to map like

$$\zeta(s) \to 1 \tag{4}$$

$$\frac{1}{\zeta(s)} \to \mu(k) \tag{5}$$

$$\frac{\zeta(2s)}{\zeta(s)} \to \lambda(k) \tag{6}$$

then we can track less standard variations by creating ratios of zeta functions:

$$\frac{\zeta(3s)}{\zeta(s)} \to A210826(k) \tag{7}$$

$$\zeta(s)^2 \to d(n) = \tau(n) \tag{8}$$

$$\frac{\zeta(2s)}{\zeta(s^2)} \to \chi_{\text{squares}}(k) \tag{9}$$

$$\frac{\zeta(q+s)}{\zeta(s^2)} \to \frac{1}{k^q} \tag{10}$$

$$\frac{\zeta(s+\frac{1}{2})}{\zeta(s+\frac{3}{2})} \to \frac{\phi(k)}{k\sqrt{k}} \tag{11}$$

$$\sqrt{\zeta(s)} = ? \tag{12}$$

Consider the duality of $\zeta(s) \to \Gamma(s)$ with inverse Mellin transform...

Dirichlet Shift Operator

We could consider an operator O^- (O^+) which shifts the argument of a zeta function by -1 ($+1$). Then use something like the product and quotient rules, to define this on Dirichlet generating functions

$$O^-[\zeta(s)] = \zeta(s-1) \tag{13}$$

$$O^-[\zeta(s)\zeta(s-1)] = \zeta(s-1)^2 + \zeta(s)\zeta(s-2) \tag{14}$$

$$O^-[\frac{\zeta(2s)}{\zeta(s)}] = \frac{\zeta(2s-1)\zeta(s) - \zeta(2s)\zeta(s-1)}{\zeta(s)^2} \tag{15}$$

we can then define the 'number theoretic derivative' of a function for this latter one we appear to have

$$\frac{\zeta(2s-1)}{\zeta(s)} \to \lambda(n) * A000188(n) \tag{16}$$

$$\frac{\zeta(2s)\zeta(s-1)}{\zeta(s)^2} \to A074722(n) \tag{17}$$

so as $\zeta(2s)/\zeta(s) \to \lambda(n)$ we could say

$$\delta\lambda(n) = \lambda(n)*A000188(n) - A074722(n) = 0, -1, -2, 0, -4, 1, -6, -4, -2, 1, -10, -4, -12, 1, -2, -2, -16, \cdots$$

where clearly, we have every prime, $\delta\lambda(p) = 1-p$, this is nice. We have depending on definition

$$\delta\mu(n) = -A007431(n) \text{ or } \mu(n) - A007431(n) \tag{18}$$

this is

$$\delta\mu(n) = -\sum_{d|n} \phi(d)\mu(\frac{n}{d})$$

and therefore

$$\phi(n) = -\sum_{d|n} \delta\mu(d)$$

then

$$\delta|\mu(n)| = \delta\frac{\zeta(s)}{\zeta(2s)} = \frac{\zeta(s-1)\zeta(2s) - \zeta(2s-1)\zeta(s)}{\zeta(2s)^2} = \frac{\zeta(s-1)}{\zeta(2s)} - \frac{\zeta(2s-1)\zeta(s)}{\zeta(2s)^2} \tag{19}$$

$$\delta|\mu(n)| = A063659(n) - H(n) \tag{20}$$

where $H(n) = 1$ unless $n = 9, 16, 18...?$, and 2 otherwise. It seems that

$$\frac{\zeta(2s-1)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\phi(k)}{(k^2)^s} = \sum_{k=1}^{\infty} \frac{\chi_{\text{square}}(k)\phi(\sqrt{k})}{k^s}$$

then there is the convolutions

$$\frac{\zeta(2s-1)\zeta(s)}{\zeta(2s)^2} = |\mu(n)| *_D \chi_{\text{square}}(n)\phi(\sqrt{n}) = \sum_{d|n} |\mu(\frac{n}{d})|\chi_{\text{square}}(d)\phi(\sqrt{d}) = \chi_{\text{square}}(n)\phi(\sqrt{n}) + (1-\chi_{\text{square}}(n)) = H(n)$$

we have

$$\delta\tau(n) = 2\sigma_1(n) \tag{21}$$

$$\delta\sigma_1(n) = n\tau(n) + \sigma_2(n) \tag{22}$$

$$\delta\sigma_2(n) = n\sigma_1(n) + \sigma_3(n) \tag{23}$$

$$\delta\sigma_k(n) = n\sigma_{k-1}(n) + \sigma_{k+1}(n) \tag{24}$$

noting that $\sigma_{-1}(n) = \sigma_1(n)/n$, also then

$$\delta n\tau(n) = 2\sigma_1(n) \tag{25}$$
$$\delta\tau(n) = 2\sigma_1(n) \tag{26}$$
$$\delta^2\tau(n) = 2n\tau(n) + 2\sigma_2(n) \tag{27}$$
$$\delta^3\tau(n) = (4+2n)\sigma_1(n) + 2\sigma_3(n) \tag{28}$$

By thinking carefully about the linearity of derivatives and implying the δ is a linear operator, then we can easily show that

$$\delta^k \log(n) = n^k \log(n) \tag{29}$$

for all integer k. In fact in general

$$\delta^k \log^l(n) = n^k \log^l(n)$$

A stunning result appears to be

$$\delta\Lambda(n) = \log(n^{\phi(n)})$$

for which the DGF is

$$\frac{\zeta(s-1)\zeta'(s)}{\zeta(s)^2} - \frac{\zeta'(s-1)}{\zeta(s)}$$

Further Operators

Consider the operator

$$\kappa = [\frac{1}{\zeta(s)}\delta\zeta(s)]$$

such that a factor of $\zeta(s)$ is applied, the derivative taken, and then the factor removed.

$$\kappa\zeta(s) = \frac{1}{\zeta(s)}\delta\zeta(s)^2 = \frac{1}{\zeta(s)}2\zeta(s)\zeta(s-1) = 2\zeta(s-1)$$

therefore

$$\kappa 1 = k$$

we also have

$$\kappa\zeta(s-1) \to \frac{\zeta(s-1)^2}{\zeta(s)} + \zeta(s-2) \tag{30}$$
$$\kappa k = P(k) + k^2 \tag{31}$$

where $P(k)$ is $A018804(k)$.

We can consider more complicated operators

$$\epsilon = 1 + \delta + \delta^2 + \delta^3 + \cdots$$

then

$$\epsilon\zeta(s) \to \zeta(s) + \zeta(s-1) + \zeta(s-2) + \cdots$$

which implies

$$\epsilon 1 = 1 + k + k^2 + k^3 + \cdots = \frac{1}{1-k}$$

Kernel Guided Transform

For the Lambert transform we have the nicve property that

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} b_n x^n$$

where the relationship is

$$b_n = \sum_{d|n} a_n$$

but we can generalise this by realising that

$$\sum_{n=1}^{\infty} a_n f(x^n) = \sum_{n=1}^{\infty} b_n x^n$$

where

$$f(x) = \sum_{n=0}^{\infty} \kappa(n) x^n$$

generates a new transform, we then know for $\kappa(n) = 1$, we have the Lambert transform.

Ratio of sigma

If we let

$$\kappa(n) = \frac{\sigma_2(n^2)}{\sigma_1(n^2)}$$

then it seems if If we let

$$\kappa(n) = \phi(n)$$

then it seems if If we let

$$\kappa(n) = \mu(n)$$

then it seems if

a_n	b_n	ζ_a	ζ_b
$\mu(n)$	$n\phi(n)$	$1/\zeta(s)$	$\frac{\zeta(s-1)}{\zeta(s)}$
$\phi(n)$	n^2	$\frac{\zeta(s-1)}{\zeta(s)}$	$\zeta(s-2)$
$\mu(n)^2$	$A034444(n) = d^*(n)$	$\zeta(s)/\zeta(2s)$	$\zeta(s)^2/\zeta(2s)$
$\sigma_1(n)$	...	$\zeta(s)\zeta(s-1)$	$\zeta(s-2)\zeta(s)^2$
n	σ_2	$\zeta(s-1)$	$\zeta(s)\zeta(s-2)$

a_n	b_n	ζ_a	ζ_b
$\phi(n)$	$\sum_{d\mid n} \phi\phi$	$\frac{\zeta(s-1)}{\zeta(s)}$	$\frac{\zeta(s-1)^2}{\zeta(s)^2}$
$\mu(n)$	$\sum_{d\mid n} \phi\mu$	$\frac{1}{\zeta(s)}$	$\frac{\zeta(s-1)}{\zeta(s)^2}$
1	n	$\zeta(s)$	$\zeta(s-1)$
n	$\sum_{k=1}^{n} gcd(n,k)$	$\zeta(s-1)$	$\zeta(s-1)^2/\zeta(s)$

a_n	b_n	ζ_a	ζ_b
$\phi(n)$	$\sum_{d\mid n} \phi\mu$	$\frac{\zeta(s-1)}{\zeta(s)}$	$\frac{\zeta(s-1)}{\zeta(s)^2}$