Fekih-Ahmed Transform

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Abstract

We consider a 'Fekih-Ahmed' transform based on a single equation which is probably a coincidence.

Main

Consider the series expansion for the reciprocal Gamma function

$$\frac{1}{\Gamma(z)} = \sum_{n=1}^{\infty} a_n z^n = z + \gamma z^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right) z^3 + \cdots \tag{1}$$

According to the Wikipedia page (Reciprocal gamma function) there is an integral formula for these coefficients due to Fekih-Ahmed:

$$a_n = \frac{(-1)^n}{\pi n!} \int_0^{\infty} e^{-t} \Im[(\log(t) - i\pi)^n] dt \tag{2}$$

excellent, looks interesting.

Let's draw analogy from the Mellin transform, and the definition of the Gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{3}$$

both of these equations have an e^{-t}, the integral domain is the same and both relate to Gamma functions. We also note the presence of the term

$$\chi(n) = \frac{(-1)^n}{n!}$$

which is critical in the Ramanujan master theorem allowing us to express the Mellin transform of a function with expansion

$$f(x) = \sum_{k=0}^{\infty} \chi(k)\phi(k)x^k$$

as

$$\mathcal{M}[f](s) = \int_0^{\infty} x^{s-1} f(x)\, dx = \Gamma(s)\phi(-s) \tag{4}$$

where possible. Based on this we absorb the coefficient in the definition and define the Fekih-Ahmed transform as

$$\mathcal{A}[f](n) = \int_0^{\infty} f(x) \Im[(\log(t) - i\pi)^n] \frac{dx}{\pi} \tag{5}$$

and allow functions to be defined by

$$g(x) = \sum_{k=1}^{\infty} \chi(k)\mathcal{A}[f](k)x^k \tag{6}$$

namely

$$\frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} \chi(k)\mathcal{A}[e^{-x}](k)z^k \tag{7}$$

Put into words: "The gamma function is Mellin transform of e^{-x}, the function whose alternating exponential coefficient is 1." and "The alternating exponential coefficient of the reciprocal gamma function is the Fekih-Ahmed transform of e^{-x}".

This is not quite symmetric, but there seems to be a rough interplay between the concepts. Let's explore further...

Examples

It seems that

$$\frac{(-1)^n}{n!}\mathcal{A}\left[\frac{1}{(x+1)^k}\right](n) = \frac{|S_1(k+1,n+1)|}{(k-1)!}, n < k$$

we are in sketchy territory for inserting random functions into the transform, as we are trying to match up terms from series expansions and from the integral evaluations. It will be more productive to take a series expansion of a related function such as

$$\frac{1}{\Gamma(z)\Gamma(z+1)} = z + 2\gamma z^2 + \cdots$$

and work out the contents of the FA transform that match this, perhaps it will be a simple function:

Following the logic of Mellin transforms, it might be best to explore hypergeometric type arguments. For example the inverse Mellin transform of $\Gamma(s)^2$ is

$$\mathcal{M}^{-1}[\Gamma(s)^2](x) = 2K_0(2\sqrt(x))$$

so we consider

$$g(x) = \sum_{k=0}^{\infty} \mathcal{A}\left[2K_0(2\sqrt{x})\right](k+1)z^k = 1+2\gamma z+2\gamma^2 z^2+\frac{4\gamma^3+2\zeta(3)}{3}z^3+\frac{2\gamma^4+2\gamma\zeta(3)}{3}z^4+\cdots = A+B+C+\cdots$$

which seems to have a component of

$$A = \sum_{n=0}^{\infty} \frac{2^n}{n!}\gamma^n z^n = e^{2\gamma z}$$

this seems to be the right kind of language to describe these expansions.

We can inspect in more detail the meaning of the term in the integral. In terms of a probability distribution or similar:

$$\frac{(-1)^1\mathcal{A}[f](1)}{1!} = \int_0^{\infty} f(x)dx$$

the $n = 1$ term is the normalization.

$$\frac{(-1)^2\mathcal{A}[f](2)}{2!} = \int_0^{\infty} -\log(x)f(x)dx$$

the $n = 2$ term is the expectation of the 'negative log'.

$$\frac{(-1)^3 \mathcal{A}[f](3)}{3!} = \int_0^\infty (\log(t)^2/2 - \zeta(2)) f(x) dx$$

$$\frac{(-1)^4 \mathcal{A}[f](4)}{4!} = \int_0^\infty (\zeta(2) \log(x) - \log(t)^3/6) f(x) dx$$

The coefficients appear to be A109447,

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