Inverse Functions Define Series (Polylogarithm)

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Introduction

I am converting this answer from a post online

Main

For the Polylogarithm we have the series representation

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

if we perform a [series reversion][1] on this (term by term) we end up with an expansion for the inverse function

$$\mathrm{Li}_s^{-1}(z) = \sum_{k=1}^{\infty} a_k z^k$$

the first few coefficients are

$$\begin{aligned} a_1 &= 1 \\ a_2 &= -2^{-s} \\ a_3 &= 2^{1-2s} - 3^{-s} \\ a_4 &= 56^{-s} - 8^{-s}(5 + 2^s) \\ &\cdots \end{aligned} \tag{1}$$

there may be a pattern in there somewhere, but the terms seem to grow quite large and complicated rather quickly. For some reason I considered looking at the inverse Mellin transform of these coefficients, multipled by a gamma function, we can denote these functions $e_k(x)$

$$e_k(x) = \mathcal{M}^{-1}[\Gamma[s]a_k(s)](s)$$

these begin

$$e_1(x) = e^{-x} \tag{2}$$

$$e_2(x) = -e^{-2x} \tag{3}$$

$$e_3(x) = 2e^{-4x} - e^{-3x} \tag{4}$$

$$e_4(x) = -5e^{-8x} + 5e^{-6x} - e^{-4x} \tag{5}$$

$$\cdots \tag{6}$$

in each term $e_k(x)$ there are $P(k-1)$ exponential functions, where $P(k)$ from $k=0$ goes like $1, 1, 2, 3, 5, 7, \cdots$ and are the partition numbers [A000041][2]. The coefficients in this grid of exponentials goes like:

$$\alpha_1 = [+1] \tag{7}$$
$$\alpha_2 = [-1] \tag{8}$$
$$\alpha_3 = [-1, 2] \tag{9}$$
$$\alpha_4 = [-1, 5, -5] \tag{10}$$
$$\alpha_5 = [-1, 6, 3, -21, 14] \tag{11}$$
$$\alpha_6 = [-1, 7, 7, -28, -28, 84, -42] \tag{12}$$
$$\alpha_7 = [-1, 8, 8, 4, -36, -72, -12, 120, 180, -330, 132] \tag{13}$$

, and appear to be given by [A111785][3]. Interestingly, the powers of the exponentials also appear to have a sequence, the terms go like [A074139][4], which is titled "Number of divisors of A036035(n).". A036035 is titled "Least integer of each prime signature, in graded (reflected or not) colexicographic order of exponents."

We can recreate a coefficient by performing the Mellin transform and dividing through by $\Gamma(s)$

$$a_k(s) = \frac{1}{\Gamma(s)}\mathcal{M}[e_k(x)](s) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1} e_k(x)\, dx$$

Then we can write

$$e_k(x) = \sum_{l=1}^{P(k-1)} \alpha_{kl} e^{-\beta_{kl} x}$$

where β_k are the anaolgous rows of A074139

$$\beta_1 = [1] \tag{14}$$
$$\beta_2 = [2] \tag{15}$$
$$\beta_3 = [3, 4] \tag{16}$$
$$\beta_4 = [4, 6, 8] \tag{17}$$
$$\beta_5 = [5, 8, 9, 12, 16] \tag{18}$$
$$\beta_6 = [6, 10, 12, 16, 18, 24, 32] \tag{19}$$
$$\tag{20}$$

we we know then the Mellin transform of $ae^{-bx} = ab^{-s}\Gamma(s)$, and it's as simple as, grouping the terms which are now understood

$$a_k(s) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1} \sum_{l=1}^{P(k-1)} \alpha_{kl} e^{-\beta_{kl} x}\, dx$$

$$a_k(s) = \sum_{l=1}^{P(k-1)} \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1} \alpha_{kl} e^{-\beta_{kl} x}\, dx$$

$$a_k(s) = \sum_{l=1}^{P(k-1)} \frac{1}{\Gamma(s)} \alpha_{kl} \beta_{kl}^{-s} \Gamma(s)$$

$$a_k(s) = \sum_{l=1}^{P(k-1)} \alpha_{kl} \beta_{kl}^{-s}$$

$$\mathrm{Li}_s^{-1}(z) = \sum_{k=1}^{\infty} a_k(s) z^k = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \alpha_{kl} \beta_{kl}^{-s} z^k$$

the limitation here is understanding where the terms in the sequences referenced come from, and finding whether any tractable forms exist for α and β. We already know that $\beta_{kl} = \sigma_0(\gamma_{kl})$ for the γ in A036035, and divisor counting function σ.

$$\mathrm{Li}_s^{-1}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \frac{\alpha_{kl} z^k}{\sigma(\gamma_{kl})^s}$$

Transform and Duality

In this sense we could define a transform between the divisor function and the polylogarithm

$$T[\sigma(n)] \rightarrow \mathrm{Li}_s(z)$$

we might ask, what other number-theory related functions transform to in this manner? The most interesting (for which there is an obvious pattern) seems to be for the Liouville lambda function,

$$T[\lambda(n)] \rightarrow (z + z^3 + z^5 + z^7 + z^9 + z^{11} + \cdots) + (-1)^s (z^2 + z^4 + z^6 + z^8 + z^{10} + \cdots) = \frac{z + (-1)^s z^2}{1 - z^2}$$

it seems that

$$\mathrm{Inv}(z - z^2) = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \alpha_{kl} z^k \mu(\gamma_{kl}) = z\mathrm{GF}(\mathrm{Catalan})$$

$$\sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \alpha_{kl} z^{k-1} \Omega(\gamma_{kl}) = z\mathrm{altGF}(\mathrm{Catalan})$$

A nice result seems to be

$$\mathrm{Inv}(\sqrt{z}\,\mathrm{atanh}(\sqrt{z}))) = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \frac{\alpha_{kl} z^k}{\sigma(\gamma_{kl}^2)}$$

$$\mathrm{Inv}(\frac{z}{n}\Phi(z, 1, \frac{1}{n})) = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \frac{\alpha_{kl} z^k}{\sigma(\gamma_{kl}^n)}$$

and in general

$$\mathrm{Inv}(\frac{z}{n^s}\Phi(z, s, \frac{1}{n})) = \sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \frac{\alpha_{kl} z^k}{\sigma(\gamma_{kl}^n)^s}$$

Nice examples are

$$\sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \alpha_{kl} z^{k-1} \Omega(\gamma_{kl})^2 = (-z + 4z^2 - 9z^3 + 16z^4 - \cdots) = A007297$$

$$\sum_{k=1}^{\infty} \sum_{l=1}^{P(k-1)} \alpha_{kl} z^{k-1} \Omega(\gamma_{kl}^n)^s = (-(n)^s z + (2n)^s z^2 - (3n)^s z^3 + (4n)^s z^4 - \cdots)$$