Polynomial Representation in Alternative Matrix Form

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Abstract

A new representation of polynomials is investigated by defining the concept of summation Σ and product Π inverses. Then by defining an augmented matrix product it is found that two polynomials can be multiplied with their components retained in a resulting matrix. This is then shown to work for compositions of polynomials by using block matrices.

1 Introduction

This paper addresses the problem $P_3(x) = P_1(x)P_2(x)$, where each P is a polynomial with (at the moment) constant powers of x. After searching for a representation a possible technique was found.

Let

$$P = x^2 + x + .$$

(1)

Here, introduce the concept of an inverse sum, to turn a string of terms such as equation 1 into a vector for example

P=(x$^2 + x+) \rightarrow v_i =$

$\begin{bmatrix} x^2 \\ x \\ \\ \end{bmatrix}$

(2)

Such that

P =

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix} \cdot$$

$$\begin{bmatrix}x^2 \\ x \\ \\ \end{bmatrix}$$

(3)

We then extrude the vector, (inverse product) as follows

$\mathrm{v}_i \to M_{ij} =$

$$\begin{bmatrix}x & x & \\ 1 & x & \\ 1 & 1 & \end{bmatrix}$$

(4)

Such that

$\mathrm{v}_i =$

$$\begin{bmatrix}x & x & \\ 1 & x & \\ 1 & 1 & \end{bmatrix} \star$$

$$\begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix}$$

(5)

where the $\star$ augmented matrix-vector product is defined as

$\mathrm{M} \star u = \prod_j M_{ij} u_j = v_i$

(6)

and the $\star$ augmented matrix-matrix product is defined as

A $\star B = \prod_j A_{ij}B_{jk} = C_{ik}$
(7)

as opposed to the regular matrix-matrix product

A $\cdot B = \sum_j A_{ij}B_{jk} = C_{ik}$
(8)

With these tools it will be possible to progress onto the multiplication of polynomials.

2 Multiplying Polynomials

For successful polynomial multiplication it was found that the right hand matrix in the $\star$ product must be the transpose of its representation matrix. For example, if there are two polynomials

P$_1 = x^2 + x +,$
(9)

$$P_2 = ax^2 + bx + c,$$

(10)

which can be mapped to their respective matrix representations

P$_1 \rightarrow A_{ij} =$
$\begin{bmatrix} x & x & \alpha \\ 1 & x & \beta \\ 1 & 1 & \gamma \end{bmatrix},$

(11)

$\mathrm{P}_2 \to B_{ij} =$

$$\begin{bmatrix} x & x & a \\ 1 & x & b \\ 1 & 1 & c \end{bmatrix},$$
(12)

then their product P_3 will be

$\mathrm{P}_3 = \sum_{i,k} A_{ij} \star B^T_{jk}.$
(13)

This can be written explicitly in matrix form as

$\mathrm{P}_3 = \sum_{i,k}$

$$\begin{bmatrix} x & x & \alpha \\ 1 & x & \beta \\ 1 & 1 & \gamma \end{bmatrix} \star$$

$$\begin{bmatrix} x & 1 & 1 \\ x & x & 1 \\ a & b & c \end{bmatrix},$$
(14)

which, keeping the positions of terms for clarity, is

$\mathrm{P}_3 = \sum_{i,k}$

$$\begin{bmatrix} xxxx\alpha a & x1xx\alpha b & x1x1\alpha c \\ 1xxx\beta a & 11xx\beta b & 11x1\beta c \\ 1x1x\gamma a & 111x\gamma b & 1111\gamma c \end{bmatrix}$$

$=$

$\sum_{i,k}$

$$\begin{bmatrix} \alpha a x^4 & \alpha b x^3 & \alpha c x^2 \\ \beta a x^3 & \beta b x^2 & \beta c x \\ \gamma a x^2 & \gamma b x & \gamma c \end{bmatrix}.$$

(15)

After the sum it can be seen that

$$\mathrm{P}_3 = \alpha a x^4 + (\alpha b + \beta a)x^3 + (\alpha c + \beta b + \gamma a)x^2 + (\beta c + \gamma b)x + \gamma c,$$

(16)

which is indeed $(x^2 + x+)(ax^2 + bx + c)$ by standard multiplication.

To obtain the representation of P_3 the process can be repeated to obtain

$\mathrm{P}_3 \rightarrow C_{ij} =$

$$\begin{bmatrix} x & x & x & x & \alpha a \\ 1 & x & x & x & \alpha b + \beta a \\ 1 & 1 & x & x & \alpha c + \beta b + \gamma a \\ 1 & 1 & 1 & x & \beta c + \gamma b \\ 1 & 1 & 1 & 1 & \gamma c \end{bmatrix}.$$

(17)

The form of the coefficient column are clearly the the inner product of the two independent coefficient vectors (right most column). This can be separated into the sum of polynomial representations

$\mathrm{P}_3 \rightarrow C_{ij} =$

$$\begin{bmatrix} x & x & x & x & \alpha a \\ 1 & x & x & x & \alpha b \\ 1 & 1 & x & x & \alpha c \\ 1 & 1 & 1 & x & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} x & x & x & x & 0 \\ 1 & x & x & x & \beta a \\ 1 & 1 & x & x & \beta b \\ 1 & 1 & 1 & x & \beta c \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} x & x & x & x & 0 \\ 1 & x & x & x & 0 \\ 1 & 1 & x & x & \gamma a \\ 1 & 1 & 1 & x & \gamma b \\ 1 & 1 & 1 & 1 & \gamma c \end{bmatrix} .$$

(18)

Any row with a coefficient of 0 can be deleted, leading to

$P_3 \rightarrow C_{ij} =$

$$\begin{bmatrix} x & x & x & x & \alpha a \\ 1 & x & x & x & \alpha b \\ 1 & 1 & x & x & \alpha c \end{bmatrix} +$$

$$\begin{bmatrix} 1 & x & x & x & \beta a \\ 1 & 1 & x & x & \beta b \\ 1 & 1 & 1 & x & \beta c \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 1 & x & x & \gamma a \\ 1 & 1 & 1 & x & \gamma b \\ 1 & 1 & 1 & 1 & \gamma c \end{bmatrix} .$$

(19)

Any column which is constant throughout the matrix can be taken out as a factor "polynomial". In this case that is two columns of x and a column of α in the first term (αx^2), one column of 1,x and β in term two, and so on, leading to

$P_3 \rightarrow C_{ij} = \alpha x^2$

$$\begin{bmatrix} x & x & a \\ 1 & x & b \\ 1 & 1 & c \end{bmatrix} + \beta x$$

$$\begin{bmatrix} x & x & a \\ 1 & x & b \\ 1 & 1 & c \end{bmatrix} + \gamma$$

$$\begin{bmatrix} x & x & a \\ 1 & x & b \\ 1 & 1 & c \end{bmatrix}.$$
(20)

Which clearly leaves a common factor of the polynomial that is represented by the matrix, that is, P_2. This equation could equally have been written in representation form however it is not yet clear how to move directly from a product of two polynomial representations to a single new (larger) representation.

It is a hope that with some clever notation the large string of x's and 1's need not be written in the future.

3 Keeping to Representation Form

The last equation could have been written in representation form

$\mathrm{P}_3 \rightarrow C_{ij} =$

$$\begin{bmatrix} x & x & \alpha \\ 1 & x & 0 \\ 1 & 1 & 0 \end{bmatrix} \star P_2^T + \begin{bmatrix} x & x & 0 \\ 1 & x & \beta \\ 1 & 1 & 0 \end{bmatrix} \star P_2^T + \begin{bmatrix} x & x & 0 \\ 1 & x & 0 \\ 1 & 1 & \gamma \end{bmatrix} \star P_2^T . (21)$$

A further reduction could be made base on the 0 valued row coefficients, however, based on the original form of the product it is clear that such $\star$ products, when summed together, can all go to form a matrix on the outside to restore the original

$\mathrm{P}_3 \rightarrow C_{ij} =$

$$\begin{bmatrix} x & x & \alpha \\ 1 & x & \beta \\ 1 & 1 & \gamma \end{bmatrix} \star \begin{bmatrix} x & x & a \\ 1 & x & b \\ 1 & 1 & c \end{bmatrix}^T . (22)$$

In another article [Ref] we consider the implications of two spare vector products, that is when a column vector is multiplied on another column vector. We defined the result as another column vector twice as high i.e

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} =$$

$$\begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

(23)

This transformation was discovered to be vital to keeping to representation form, it is also a short hand for multiplication! Using a 2x2 example we seek the transformation

$$\begin{bmatrix} x & a \\ 1 & b \end{bmatrix} *$$

$$\begin{bmatrix} x & c \\ 1 & d \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x & x & ac \\ 1 & x & ad \\ 1 & x & bc \\ 1 & 1 & bd \end{bmatrix} =$$

$$\begin{bmatrix} x & x & ac \\ 1 & x & ad + bc \\ 1 & 1 & bd \end{bmatrix}$$

(24)

This is directly achieved by separating the columns of the two 2x2 matrices and taking this strange product.

$$\begin{bmatrix} x & a \\ 1 & b \end{bmatrix} *$$

$$\begin{bmatrix} x & c \\ 1 & d \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} & \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} x^2 & ac \\ x & ad \\ x & bc \\ 1 & bd \end{bmatrix}$$

(25)

where we can fold the expansion over x's into a column vector alongside the coefficient vector. This process is quicker than before and avoids using the transpose of a matrix. It has now become clear we do not need to fully expand out the x and 1 terms in the polynomial matrix at all times.

4 Regular Multiplications

We can in addition utilize regular matrix multiplications, however the 'rules' have changed with this form. For example, the identity is still

$$IP= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & a \\ 1 & b \end{bmatrix} =P \quad (26)$$

But in order to multiply a polynomial by a factor of α we must use the scaling matrix $\sqrt{\alpha}I$. As

$$\beta IP = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x & a \\ 1 & b \end{bmatrix} =\beta^2 P \quad (27)$$

Bearing in mind that decoupling the kth order polynomial from its matrix representation takes the form

$$o_k^T \cdot P_k \star o_k \quad (28)$$

where o_k is has k elements which are 1.

5 Differentiation of a Polynomial

It must then be possible to seek a transformation to representation matrices that describes the process of differentiation, let

P= $\alpha x^3 + \beta x^2 + \gamma x + \delta$
(29)

Then

dP $\overline{dx = 3\alpha x^2 + 2\beta x + \gamma}$
(30)

Which could be represented by the transformation

$$\begin{bmatrix} x & x & x & \alpha \\ 1 & x & x & \beta \\ 1 & 1 & x & \gamma \\ 1 & 1 & 1 & \delta \end{bmatrix} \rightarrow$$
$$\begin{bmatrix} 1 & x & x & 3\alpha \\ 1 & 1 & x & 2\beta \\ 1 & 1 & 1 & 1\gamma \\ 1 & 1 & \frac{1}{x} & 0\delta \end{bmatrix} =$$
$$\begin{bmatrix} x & x & 3\alpha \\ 1 & x & 2\beta \\ 1 & 1 & 1\gamma \end{bmatrix}$$
(31)

It is clear the size of a matrix shrinks when differentiated, corresponding to a loss of information! Integration would increase the size of a matrix, with the same issue of new information being generated like the constant of integration.

6 Compositions of Polynomials

Products can also be taken with compositions of polynomials take for example

P(x)= x$^2 + x + 1 \rightarrow$
$$\begin{bmatrix} x & x & 1 \\ 1 & x & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Q(x)= x + 2 $\rightarrow$

$$\begin{bmatrix} x & 1 \\ x & 2 \end{bmatrix}$$

R(x)= 2x + 3 $\rightarrow$

$$\begin{bmatrix} x & 2 \\ 1 & 3 \end{bmatrix}$$

If we require the product of $P\big(Q(x)\big)R(x)$ that can be displayed as

$$P\big(Q(x)\big)R(x) \rightarrow$$

$$\begin{bmatrix} \begin{bmatrix} x & 1 \\ x & 2 \end{bmatrix} & \begin{bmatrix} x & 1 \\ x & 2 \end{bmatrix} & 1 \\ 1 & \begin{bmatrix} x & 1 \\ x & 2 \end{bmatrix} & 1 \\ 1 & 1 & 1 \end{bmatrix} \star$$

$$\begin{bmatrix} x & 1 & 1 \\ x & x & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

(32)

Recall the transponse was taken for the representation of R.

7 Series Representation

It is possible to construct an infinite matrix to represent a series and also take products of such matrices. Taking the series form of $exp(x)$

$\mathrm{e}^x = \lim_{M\rightarrow\infty} \sum_{i=0}^{i=M} \frac{x^i}{i!}$

(33)

the associated representation matrix will be

$\mathrm{e}^x \rightarrow \lim_{M\rightarrow\infty}$

$$\begin{bmatrix} x & x & \cdots & x & \frac{1}{M!} \\ 1 & x & \cdots & x & \frac{1}{(M-1)!} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & x & \frac{1}{1!} \\ 1 & 1 & \cdots & 1 & \frac{1}{0!} \end{bmatrix}$$
(34)

Then the product can be displayed

$$\mathrm{e}^x e^x \to \lim_{(M,N)\to\infty}$$
$$\begin{bmatrix} x & x & \cdots & x & \frac{1}{M!} \\ 1 & x & \cdots & x & \frac{1}{(M-1)!} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \cdots & x & \frac{1}{1!} \\ 1 & 1 & \cdots & 1 & \frac{1}{0!} \end{bmatrix} \star$$
$$\begin{bmatrix} x & 1 & \cdots & 1 & 1 \\ x & x & \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x & x & \cdots & x & 1 \\ \frac{1}{N!} & \frac{1}{(N-1)!} & \cdots & \frac{1}{1!} & \frac{1}{0!} \end{bmatrix}$$
(35)

Which results in

$$\Leftrightarrow \sum$$
$$\begin{bmatrix} \frac{x^{M+N}}{M!N!} & \frac{x^{M+N-1}}{M!(N-1)!} & \cdots & \frac{x^{M+1}}{M!1!} & \frac{x^{M}}{M!0!} \\ \frac{x^{M-1+N}}{(M-1)!N!} & \frac{x^{M-1+N-1}}{(M-1)!(N-1)!} & \cdots & \frac{x^{M-1+1}}{(M-1)!1!} & \frac{x^{M-1}}{(M-1)!0!} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{x^{1+N}}{1!N!} & \frac{x^{1+N-1}}{1!(N-1)!} & \cdots & \frac{x^{2}}{1!1!} & \frac{x}{1!0!} \\ \frac{x^{N}}{0!N!} & \frac{x^{N-1}}{0!(N-1)!} & \cdots & \frac{x}{0!1!} & \frac{1}{0!0!} \end{bmatrix}$$
(36)

Which is collected into the left hand side and undoing the inverse summation leads to the right hand side

$\Leftrightarrow$

$$\begin{bmatrix} \sum_{j=0}^{j=N} \frac{x^{M+j}}{M!j!} \\ \sum_{j=0}^{j=N} \frac{x^{(M-1)+j}}{(M-1)!j!} \\ \vdots \\ \sum_{j=0}^{j=N} \frac{x^{1+j}}{1!j!} \\ \sum_{j=0}^{j=N} \frac{x^{j}}{0!j!} \end{bmatrix}$$

$\Leftrightarrow \sum_{i=0}^{i=M} \sum_{j=0}^{j=N} \frac{x^{i+j}}{i!j!}$

(37)

8 Curiosities

A polynomial with value $1, 0$ or x could be written

$1 \rightarrow$

$$\begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix}$$

$0 \rightarrow$

$$\begin{bmatrix} x & 0 \\ 1 & 0 \end{bmatrix}$$

x $\rightarrow$

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \mathrm{x}^T$$

Therefore

$1 = 1 \cdot 1 \rightarrow$

$$\begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix} \star$$

$$\begin{bmatrix} x & 1 \\ 0 & 1 \end{bmatrix} = \sum$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(38)

Or

$$1 \rightarrow$$

$$\begin{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} x & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} x & 0 \\ 1 & 1 \end{bmatrix} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x & 1 & x & 0 \\ 1 & 0 & 1 & 0 \\ x & 0 & x & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(39)

Take the $\star$ product

$$1 = \sum$$

$$\begin{bmatrix} x & 1 & x & 0 \\ 1 & 0 & 1 & 0 \\ x & 0 & x & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \star$$

$$\begin{bmatrix} x & 1 & x & 1 \\ 1 & 0 & 0 & 1 \\ x & 1 & x & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \sum$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(40)

Another curiosity could be to write in sqrt powers of x

$$\begin{bmatrix} \sqrt{x} & \sqrt{x} & \sqrt{x} & \sqrt{x} & a \\ 1 & 1 & \sqrt{x} & \sqrt{x} & b \\ 1 & 1 & 1 & 1 & c \end{bmatrix}$$

(41)