Constants Whose Engel Expansions are the k-rough Numbers

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1 Abstract

I find series for constants whose Engel Expansions are equal to the integer sequences known as k-rough numbers. They appear to be expressed as a finite (but large) sum of hypergeometric functions for the general case.

2 Main

2.1 Engel Expansion

The Engel expansion is the weakly increasing set of integers $n_1, n_2, n_3, n_4, \cdots$ such that for a real number x

$$x = \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \frac{1}{n_1 n_2 n_3 n_4} + \cdots$$

(1)

this gives an attachment of certain integer sequences to constants. The integer sequence can be generated uniquely by the constant with a simple algorithm.

2.2 Rough numbers

We can define the k-rough numbers as the set of integers not divisible by all the primes below k. Generally k is picked to be a prime.

2.3 5-Rough Numbers

The five rough numbers (OEIS A007310) have generating function (indexed from 0)

$$G(x) = \frac{(1 + 4x + x^2)}{(x+1)(1-x)^2}$$

(2)

the inverse-Z-transform of $G(1/x)$ is

$$\mathcal{Z}^{-1}[G(1/x)](n) = \frac{1}{2\pi i}\oint_c G\left(\frac{1}{x}\right) x^{n-1}\, dx \tag{3}$$

$$\mathcal{Z}^{-1}[G(1/x)](n) = \frac{1}{2}(3 + (-1)^{1+n} + 6n) \tag{4}$$

which describes the sequence (indexed from $n = 0$). I have phrased it in this manner as it is easier to find the generating functions of k-rough numbers than the sequence function $a(n)$. This means the constant whose Engel expansion is this sequence is

$$C_{5r} = \sum_{k=0}^{\infty} \frac{1}{\prod_{n=0}^{k} \mathcal{Z}^{-1}[G(1/x)](n)}$$

(5)

by definition. This may be rewritten as

$$C_{5r} = \pi \sum_{k=0}^{\infty} \sum_{n=1}^{2} \frac{2^{1-n-2k} \cdot 3^{-n-2k}}{\prod_{m=1}^{2} \Gamma\left(\frac{p_{1+n+m}}{p_2\#} + k\right)}$$

(6)

with primes p_k, the primorials are defined by

$$p_n\# = \prod_{k=1}^{n} p_k$$

(7)

C_{5r} may quickly be evaluated using hypergeometric functions in Mathematica. We have

$$C_{5r} = {}_1F_2\left(1; \frac{5}{6}, \frac{7}{6}; \frac{1}{36}\right) + \frac{1}{5}\, {}_1F_2\left(1; \frac{7}{6}, \frac{11}{6}; \frac{1}{36}\right)$$

(8)

2.4 7-Rough Numbers

we may repeat the same process for the 7 rough numbers (OEIS A007775), and obtain the constant

$$C_{7r} = \pi^4 \sum_{k=0}^{\infty} \sum_{n=1}^{8} \frac{2^{4-n-8k} \cdot (3 \cdot 5)^{-n-8k}}{\prod_{m=1}^{8} \Gamma\left(\frac{p^{(7)}_{2+n+m}}{p_3\#} + k\right)}$$

(9)

where here we require a special function

$$p^{(7)}_k = \begin{cases} p_k, & k < 16 \\ 7^2, & k = 16 \\ p_{k-1}, & k > 16 \end{cases}$$

(10)

we see some ability to generalize here. It turns out we may use the definition of the 7-rough numbers, like primes, if we index from 1, then let $a_7(n)$ be the n^{th} 7-rough number, the all of the terms from $a_7(2)$ to $a_7(15)$ are the primes, in general all the terms up to the square of the roughness number (i.e. 7) will be prime. So we may replace the strange piecewise function by using the expression

$$C_{7r} = \pi^4 \sum_{k=0}^{\infty} \sum_{n=1}^{8} \frac{2^{4-n-8k} \cdot (3 \cdot 5)^{-n-8k}}{\prod_{m=1}^{8} \Gamma\left(\frac{a_7(n+m)}{p_3\#} + k\right)}$$

(11)

2.5 11-Rough Numbers

There are constants, the exponent of π, which appears to be half the upper limit of the sum and product. We can label the exponent of π, ν, and write a guess at what the next constant for the 11 rough numbers might be

$$C_{11r} =? \pi^{\nu} \sum_{k=0}^{\infty} \sum_{n=1}^{2\nu} \frac{2^{\nu-n-2\nu k} \cdot (3 \cdot 5 \cdot 7)^{-n-2\nu k}}{\prod_{m=1}^{2\nu} \Gamma\left(\frac{a_{11}(n+m)}{p_4\#} + k\right)}$$

(12)

we see the largest power of x in the numerator of the generating function for the 7-rough numbers was 8. So perhaps this is 2ν, then for the 11 rough numbers, we should have $\nu = 24$, by their generating function. This turns out to be the correct expression with the rules described below. 'Guessing' the 13 rough numbers also appears to work, so this general formula has some truth to it.

2.6 ν for a given k

What are the ν, I previously wrote about how to find the generating functions of the larger k rough numbers, and they grow quickly. There I postulated that the sequence

$$\varphi(p_n\#) \to 1, 2, 8, 48, 480, 5670, 92160, \cdots$$

(13)

described the ν. Here φ is the euler totient function. If that were the case then we have for any set of prime rough numbers the general expressions

$$C_{p_k} = (2\pi)^{\frac{\varphi(p_k\#)}{2}} \sum_{x=0}^{\infty} \sum_{n=1}^{\varphi(p_k\#)} \frac{(p_k\#)^{-n-\varphi(p_k\#)x}}{\prod_{m=1}^{\varphi(p_k\#)} \Gamma\left(\frac{a_{p_k}(n+m)}{p_k\#} + x\right)}$$

(14)

and in some sense, as $p_k \to \infty$, by the definition of the k rough numbers we have

$$C_\infty = 1$$

(15)

which fulfills the requirement that 1 is not divisible by any prime. The sum in x converges extremely rapidly. only 3 terms are needed for 50 decimal places for p_3. The first term $x = 0$ only corresponds to the constant whose Engel expansion is $1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59,$ which terminates there. This is a rational number.

3 Barnes Integral

If we rewrite

$$C_{p_k} = (2\pi)^{\frac{\varphi(p_k\#)}{2}} \sum_{n=1}^{\varphi(p_k\#)} (p_k\#)^{-n} \sum_{x=0}^{\infty} \frac{(p_k\#)^{-\varphi(p_k\#)x}}{\prod_{m=1}^{\varphi(p_k\#)} \Gamma\left(\frac{a_{p_k}(n+m)}{p_k\#} + x\right)}$$

(16)

and compare to the Barnes integral defining the hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \, ds.$$

(17)

it is not ureasonable to suggest each constant will be a weigthed sum of $\varphi(p_k\#)$ hypergeometric functions, of poise ${}_1F_{\varphi(p_k\#)}$

4 Including Previous Primes

There is a constant whose Engel expansion is the set of prime numbers. (Sometimes 1 is included at the beginning as well). The expansions above slowly lose primes. Then the limit of the constant is 1. If we left those primes there then the limit of the constant would be that with the prime engel expansion. To recreate this we need to tamper with the generating functions of the rough numbers and add in the primes, then put the through the sequencing process. For example:

$$\frac{1+x+x^3+x^4+2x^5}{(x-1)^2(1+x)} = 1+2x+3x^2+5x^3+7x^4+11x^5+13x^6+17x^7+\cdots$$

(18)

this one gives

$$c = \pi \sum_{k=0}^{\infty} \frac{2^{1-2k}9^{-k}}{\Gamma(-1/6+k)\Gamma(1/6)+k} + \frac{3^{-1-2k}4^{-k}}{\Gamma(1/6+k)\Gamma(5/6+k)}$$

(19)